Working Paper Series

HEARIN CENTER FOR ENTERPRISE SCIENCE

HCES-05-02

A Unified Modeling and Solution Framework For Combinatorial Optimization Problems

By
Gary Kochenberger
Fred Glover
Bahram Alidaee
Cesar Rego

The University of Mississippi

Director, Keith Womer
School of Business Administration
The University of Mississippi
Post Office Box 1848
University, MS 38677-1848
(662) 915-5820
http://hces.bus.olemiss.edu

A Unified Modeling and Solution Framework For Combinatorial Optimization Problems

Gary Kochenberger [a], Fred Glover [b], Bahram Alidaee [c], Cesar Rego [c]

[a] University of Colorado At Denver, Denver, CO 80217-3364, USA.
gary.kochenberger@cudenver.edu

[b] Leeds School of Business, University of Colorado, Boulder, CO 80309-0419, USA.
fred.glover@colorado.edu

[c] Hearin Center for Enterprise Science, School of Business Administration, University of Mississippi, MS 38677, {balidaee, crego}@bus.olemiss.edu

Abstract — Combinatorial optimization problems, most often too complex for exact methods that guarantee optimality, are commonly approached via heuristic methods of one kind or another. The standard procedure is to craft a tailored heuristic method to suit the particular characteristics of the problem at hand, exploiting to the extent possible the structure available to enable a fruitful and efficient search process. Such methods, due to their construction, typically have limited usefulness in other problems domains. An alternative to this problem specific solution approach is a more general methodology that recasts a given problem into a common modeling format, permitting solutions to be derived by a common, rather than tailor-made, heuristic method. This paper reports on one such promising alternative.

1. Introduction:

The optimization folklore strongly emphasizes the unproductive consequences of converting problems from a specific class to a more general representation, since the "domain-specific structure" of the original setting then becomes invisible and can not be exploited by a method for the more general problem representation. Nevertheless, there is a strong motivation to attempt such a conversion in many applications to avoid the necessity to develop a new method for each new class. We demonstrate the existence of a general problem representation that frequently overcomes the limitation commonly ascribed to such models. Contrary to expectation, when a specially structured problem is translated into this general form, it often does not become much harder to solve, and sometimes becomes even easier to solve — provided the right type of solution approach is applied.

Our research over the past few years has revealed that this unified approach is not only plausible, but is surprisingly successful for a wide range of important problems, often surpassing the performance of established special-purpose methods for particular problem classes. As such, this unified approach holds great promise as a practical method for solving these important problems.

The model with this appealing property is the unconstrained binary quadratic binary programming problem, accompanied by the device of introducing quadratic infeasibility penalty functions to handle constraints. Not only is this model capable of representing a wide range of "special case" problem classes, but it can be advantageously exploited by adaptive memory (tabu search) metaheuristics and associated evolutionary (scatter search) methods. Computational outcomes disclose the effectiveness of this combined modeling and solution approach for problems from a diverse collection of challenging settings.

The Unified Model:

The unconstrained quadratic program can be written in the form:

UQP: $\min f(x) = xQx$

where Q is an n by n matrix of constants and x is an n-vector of binary variables. UQP is notable for its ability to represent a significant variety of important problems. The applicability of this representation has been reported in diverse settings such as social psychology (Harary [19]), financial analysis (Laughunn, [23], McBride and Yormak, [25]), computer aided design (Krarup and Pruzan [22]), traffic management (Gallo et al. [7], Witsgall, [32]), machine scheduling (Alidaee, Kochenberger, and Ahmadian, [1]), cellular radio channel allocation (Chardaire and Sutter [6]), and molecular conformation (Phillips and Rosen [30]). Moreover, many combinatorial optimization problems pertaining to graphs such as determining maximum cliques, maximum cuts, maximum vertex packing, minimum coverings, maximum independent sets, and maximum independent weighted sets are known to be capable of being formulated by the UQP problem as documented in papers of Pardalos and Rodgers [28], and Pardalos and Xue [29].

The application potential of UQP is yet substantially greater than this, however, due to reformulation methods that enable certain constrained models to be re-cast in the form of UQP. Hammer and Rudeanu [16] and Hansen [17] show that any quadratic (or linear) objective in bounded integer variables and constrained by linear equations can be reformulated as a UQP model. Our purpose is to report results that disclose this wide array of potential reformulations into the UQP format is not merely a representational novelty, but is a source of practical consequences. The following material draws upon recent findings in by Kochenberger, Glover, Alidaee, and Amini [21] and in Glover, Kochenberger, Alidaee and Amini [12].

Transformations:

We take as our starting point the constrained problem

$$\min x_0 = xQx$$

subject to

Ax = b, x binary

This model describes both the quadratic and linear case since the linear case results when Q is a diagonal matrix. Problems with inequality constraints can also be put into this form by representing their bounded slack variables by a binary expansion. These constrained quadratic optimization models are converted into equivalent UQP models by adding a quadratic infeasibility penalty function to the objective function in place of explicitly imposing the constraints *Ax = b.*

$$x_0 = xQx + P(Ax - b)^t(Ax - b)$$

$$= xQx + xDx + c$$
$$= x\hat{Q}x + c$$

where the matrix D and the additive constant c result directly from the matrix multiplication indicated. Dropping the additive constant, the equivalent unconstrained version of our constrained problem becomes

$$UQP(PEN): \min x\hat{Q}x,\ x\ binary$$

From a theoretical standpoint, a suitable choice of the penalty scalar P can always be chosen such that the optimal solution to UQP(PEN) is the optimal solution to the original constrained problem. (Hammer and Rudeanu [16]). Similar theoretical outcomes apply to many types of representations other than the UQP model, of course, and the issue of interest is whether there is any practical merit in undertaking such a transformation in the UQP case. The same question arises by reference to another transformation, which likewise falls within the context of the UQP model.

We refer to the preceding general transformation as transformation # 1. A very important special class of constraints that arise in many applications can be handled by an alternative approach, given below, which we call transformation #2.

Many problems have considerations that isolate two specific alternatives and prohibit both from being chosen. That is, for a given pair of alternatives, one or the other but not both *may* be chosen. If x_j and x_k are binary variables denoting whether or not alternatives j and k are chosen, the standard constraint that allows one choice but precludes both is:

$$x_j + x_k \le 1$$

Then, for a positive scalar P, adding the penalty function $Px_j x_j$ to the objective function is a simple alternative to imposing the constraint is a traditional manner. This penalty function has sometimes been used by to convert certain optimization problems on graphs (e.g., the maximum clique problem) into an equivalent UQP model. Its potential application, however, goes far beyond graph problems as we demonstrate in later sections of this paper.

2. Transformation to xQx:

We take as our starting point the constrained problem

$$\min x_0 = xQx$$

subject to

$$Ax = b, \ \ x \ binary$$

This model accommodates both quadratic and linear objective functions since the linear case results when Q is a diagonal matrix (observing that $x_j^2 = x_j$ when x_j is a 0-1 variable). Problems with inequality constraints can also be put into this form by representing their bounded slack variables by a binary expansion. These constrained quadratic optimization models are converted into equivalent UQP models by adding a quadratic infeasibility penalty function to the objective function in place of explicitly imposing the constraints $Ax = b$.

Specifically, for a positive scalar P, we have

$$x_0 = xQx + P(Ax-b)^t(Ax-b)$$

$$= xQx + xDx + c$$

$$= x\hat{Q}x + c$$

where the matrix D and the additive constant c result directly from the matrix multiplication indicated. Dropping the additive constant, the equivalent unconstrained version of our constrained problem becomes

$$UQP(PEN): \min x\hat{Q}x, \ x \ binary$$

From a theoretical standpoint, a suitable choice of the penalty scalar P can always be chosen so that the optimal solution to UQP(PEN) is the optimal solution to the original constrained problem (Hammer and Rudeanu [17]).

We refer to the preceding general transformation as *transformation #1*. A very important special class of constraints that arise in many applications can be handled by an alternative approach, given below, which we call *transformation #2*.

In particular, consider problems with considerations that isolate two specific alternatives and prohibit both from being chosen. That is, for a given pair of alternatives, one or the other but not both *may* be chosen. If x_j and x_k are binary variables denoting whether or not alternatives j and k are chosen, the standard constraint that allows one choice but precludes both is:

$$x_j + x_k \le 1$$

Then, for a positive scalar P, adding the penalty function Px_jx_k to the objective function is a simple alternative to imposing the constraint is a traditional manner. This penalty function has sometimes been used by to convert certain optimization problems on graphs (e.g., the maximum clique problem) into an equivalent UQP model [33]. Its potential application, however, goes far beyond these settings as demonstrated in this paper and in [24]. Note that variable upper bound constraints of the form $x_{ij} \le y_i$ can be accommodated by transformation # 2 by first replacing the y_i variables by their complement. The opportunity to employ this modeling "trick" in the context of transformation # 2 commonly arises in fixed charge and a variety of other problems.

3. The K-Colorable Problem:

Vertex coloring problems seek to assign colors to nodes of a graph such that adjacent nodes are assigned different colors. The k-colorable problem attempts to find such a coloring using exactly k colors. This problem is known to be NP-hard.

K-colorable problems can be modeled as satisfiability problems using the assignment variables:

Let x_{ij} to be 1 if node i is assigned color j, and to be 0 otherwise.

Since each node must be colored, we have

$$\sum_{j=1}^{K} x_{ij} = 1 \quad i = 1, n \tag{1}$$

where n is the number of nodes in the graph. To yield a feasible coloring, we must make sure that adjacent nodes are assigned different colors. This is accomplished by imposing the constraints

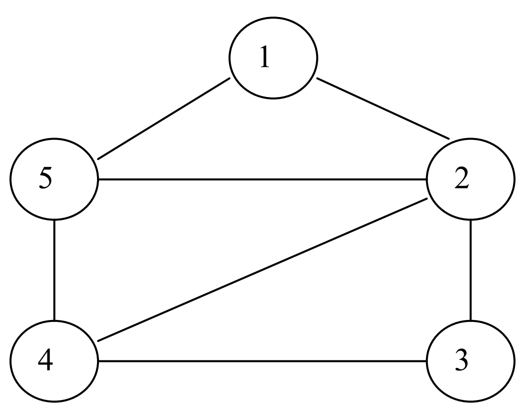
$$x_{ip} + x_{jp} \le 1 \quad p = 1, K \tag{2}$$

for all adjacent nodes (i,j) in the graph.

This problem can be re-cast into the form of UQP by using transformation # 1 on the assignment constraints of (1) and transformation #2 on the adjacency constraints of (2). Note that no new variables are required. Note also that since the model of (1) and (2) has no explicit objective function, any positive value for the penalty, P, will do. The following example gives a concrete illustration of the re-formulation process.

Example: (3-colorable)

Consider the following graph and assume we want find a feasible coloring of the nodes using 3 colors.



Our satisfiablity problem is that of finding a solution to:

$$x_{i1} + x_{i2} + x_{i3} = 1 \qquad i = 1, 5 \qquad (3)$$

$$x_{ip} + x_{jp} \le 1 \qquad p = 1, 3 \qquad (4)$$

(for all adjacent nodes i and j)

In this traditional form, the model has 15 variables and 26 constraints. To recast this problem into the form of UQP, we use transformation #1 on the equations of (3) and transformation #2 on the inequalities of (4). Arbitrarily choosing the penalty P to be 4, we get the equivalent problem:

$$UQP(Pen): \min x\hat{Q}x$$

where the additive constant is 20 and $\hat{Q}$ is:

$$\hat{Q} = \begin{bmatrix} -4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 4 & -4 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 4 & 4 & -4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 0 & -4 & 4 & 4 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 4 & 0 & 4 & -4 & 4 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 4 & 4 & -4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 & 0 & -4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 4 & -4 & 4 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & -4 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & -4 & 4 & 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 4 & -4 & 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 4 & 4 & -4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & -4 & 4 & 4 \\ 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & -4 & 4 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & -4 \end{bmatrix} i$$

Solving this unconstrained model, $x\hat{Q}x$, yields the feasible coloring:

$$x_{11}, x_{22}, x_{33}, x_{41}, x_{53} = 1 \quad all\ other\ \ x_{ij} = 0$$

Examples:

Before highlighting a variety of problem classes to which we have successfully applied the foregoing transformation approaches, we give two small examples from classical problem settings to provide concrete illustrations:

Example 1: Set Partitioning.

$$\min x_0 = 3x_1 + 2x_2 + x_3 + x_4 + 3x_5 + 2x_6$$

subject to

$$x_1 + x_3 + x_6 = 1$$
$$x_2 + x_3 + x_5 + x_6 = 1$$
$$x_3 + x_4 + x_5 = 1$$
$$x_1 + x_2 + x_4 + x_6 = 1$$

and x binary. Applying transformation #1 with P = 10 gives the equivalent UQP model:

$$UQP(PEN): \min x\hat{Q}x,\ x\ binary$$

where the additive constant, c, is 40 and

$$\hat{Q} = \begin{bmatrix} -17 & 10 & 10 & 10 & 0 & 20 \\ 10 & -18 & 10 & 10 & 10 & 20 \\ 10 & 10 & -29 & 10 & 20 & 20 \\ 10 & 10 & 10 & -19 & 10 & 10 \\ 0 & 10 & 20 & 10 & -17 & 10 \\ 20 & 20 & 20 & 10 & 10 & -28 \end{bmatrix}$$

Solving UQP(PEN) by the method of Glover et al. [11][1] we obtain an optimal solution $x_1 = x_5 = 1$ for which $x_0 = 6$. In the straightforward application of transformation #1 to this example, it is to be noted that the replacement of the original problem formulation by the UQP(PEN) model did not involve the introduction of new variables. In many applications, transformation #1 and transformation #2 can be used in concert to produce an equivalent UQP model, as demonstrated next.

Example 2: P-Median Problem:

The P-Median problem can be modeled as:

$$\min x_0 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

[1] Almost any method will work for this simple example.

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, m$$

$$\sum_{j=1}^{n} y_j = p$$

$$x_{ij} \le y_j \; \textit{for all } (i, j) \textit{ pairs}$$

where c_{ij} is the weighted distance from facility i to demand node j, $y_j = 1$ if a facility is located at location j, and $x_{ij} = 1$ if demand node i is assigned to the facility at location j. The first two sets of constraints can clearly be accommodated by transformation #1. The last set of constraints can be handled by transformation #2 by a "trick" of replacing the y variables by their compliments. (This same approach can be employed to model many fixed charge problems.)

To illustrate, consider the 12 variable example with m = n = 3, p = 2 and the C matrix

$$C = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

For P = 20, the additive constant c is 80 and the $\hat{Q}$ matrix for the equivalent UQP model is

$$\hat{Q} = \begin{bmatrix}
-20 & 20 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \\
20 & -18 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 \\
20 & 20 & -17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\
0 & 0 & 0 & -18 & 20 & 20 & 0 & 0 & 0 & 10 & 0 & 0 \\
0 & 0 & 0 & 20 & -20 & 20 & 0 & 0 & 0 & 0 & 10 & 0 \\
0 & 0 & 0 & 20 & 20 & -19 & 0 & 0 & 0 & 0 & 0 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & -17 & 20 & 20 & 10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 20 & -19 & 20 & 0 & 10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 20 & 20 & -20 & 0 & 0 & 10 \\
10 & 0 & 0 & 10 & 0 & 0 & 10 & 0 & 0 & -20 & 20 & 20 \\
0 & 10 & 0 & 0 & 10 & 0 & 0 & 10 & 0 & 20 & -20 & 20 \\
0 & 0 & 10 & 0 & 0 & 10 & 0 & 0 & 10 & 20 & 20 & -20
\end{bmatrix}$$

Solving UQP(PEN) gives $x_1 = x_6 = x_9 = y_1 = y_3 = 1$ for which $x_0 = 1$, which is optimal for the original problem.

4. Solution Approaches:

Due to its computational challenge and application potential, UQP has been the focus of a considerable number of research studies in recent years, including both exact and heuristic solution approaches. Notable recent studies addressing UQP are those by Williams [31], Pardalos and Rodgers [27], Boros, Hammer and Sun [5], Chardaire and Sutter [6], Glover, Kochenberger and Alidaee [14], Glover, Kochenberger, Alidaee, and Amini [11], Alkhamis, Hasan and Ahmed [2], Beasley [4], Lodi, Allemand and Liebling [24], Amini, Alidaee and Kochenberger [3], and Glover, Amini, Kochenberger and Alidaee [3]. Other promising work is reported by Katayama, Tani and Narihisa [20] and Merz and Freisleben [26]. These various studies approach the problem by branch and bound, decomposition, tabu search, simulated annealing, and evolutionary methods such as genetic algorithms and scatter search. Each of these approaches exhibits some degree of success. However, the exact methods degrade rapidly with problem size, and have meaningful application to general UQP problems with no more than 100 variables. For larger problems, heuristic methods are required. Two methods we have found to be particularly successful for a wide variety of problems are based on tabu search and on the related evolutionary strategy scatter search [3]. In the following we highlight our tabu search approach.

4.1 Tabu Search Overview:

Our TS method for UQP is centered around the use of strategic oscillation, which constitutes one of the primary strategies of tabu search. The variant of strategic oscillation we employ may be sketched in overview as follows.

The method alternates between constructive phases that progressively set variables to 1 (whose steps we call "add moves") and destructive phases that progressively set variables to 0 (whose steps we call "drops moves"). To control the underlying search process, we use a memory structure that is updated at *critical events,* identified by conditions that generate a subclass of locally optimal solutions. Solutions corresponding to critical events are called *critical solutions.*

A parameter *span* is used to indicate the amplitude of oscillation about a critical event. We begin with *span* equal to 1 and gradually increase it to some limiting value. For each value of *span,* a series of alternating constructive and destructive phases is executed before progressing to the next value. At the limiting point, *span* is gradually decreased, allowing again for a series of alternating constructive and destructive phases. When *span* reaches a value of 1, a *complete span cycle* has been completed and the next cycle is launched.

Information stored at critical events is used to influence the search process by penalizing potentially attractive add moves (during a constructive phase) and inducing drop moves (during a destructive phase) associated with assignments of values to variables in recent critical solutions. Cumulative critical event information is used to introduce a subtle long term bias into the search process by means of additional penalties and inducements similar to those discussed above. A complete description of the framework for the method is given in Glover, Kochenberger, Alidaee and Amini [11].

5. Computational Experience:

Our results of applying the tabu search and associated scatter search metaheuristics to combinatorial problems recast in UQP form have uniformly attractive in terms of both solution quality and computation times. As intimated earlier, although our methods are designed for the completely general form of UQP, without any specialization to take advantage of particular types of problems reformulated in this general representation, our outcomes have typically proved competitive with or even superior to those of specialized methods designed for the specific problem structure at hand. Our broad base of experience with UQP as a modeling and solution framework includes a substantial range of problem classes including

-Quadratic Assignment Problems
-Capital Budgeting Problems
-Multiple Knapsack Problems
-Task Allocation Problems (distributed computer systems)
-Maximum Diversity Problems
-P-Median Problems
-Asymmetric Assignment Problems
-Symmetric Assignment Problems
-Side Constrained Assignment Problems
-Quadratic Knapsack Problems
-Constraint Satisfaction Problems (CSPs)
-Set Partitioning Problems
-Fixed Charge Warehouse Location Problems
-Maximum Clique Problems
-Maximum Independent Set Problems
-Maximum Cut Problems
-Graph Coloring Problems
-Graph Partitioning Problems

Details of our experience with these and other problems are documented in the paper by Kochenberger, Glover, Alidaee, and Amini [21]. We are currently solving problems via UQP with more than 10,000 variables in the quadratic representation. The significance of this is underscored by that fact that the well-known transformation of the binary quadratic representation into a binary linear programming representation produces problems containing more than 50,000,000 zero-one variables. Currently we are working on enhancements that will permit larger instances to be solved.

6. Summary

We have demonstrated how a variety of disparate combinatorial problems can be solved by first re-casting them into the common modeling framework of the unconstrained quadratic binary program. Once in this unified form, the problems can be solved effectively by adaptive memory tabu search metaheuristics and associated evolutionary (scatter search) procedures.

Our findings challenge the conventional wisdom that places high priority on preserving linearity and exploiting specific structure. Although the merits of such a priority are well-founded in many cases, the UQP domain appears to offer a partial exception. In forming UQP(PEN), we destroy any linearity that the original problem may have exhibited. Moreover, any exploitable structure that may have existed originally is "folded" into the

$\hat{Q}$ matrix, and the general solution procedure we apply takes no advantage of it. Nonetheless, our solution outcomes have been remarkably successful, yielding results that rival the effectiveness of the best specialized methods.

This combined modeling/solution approach provides a unifying theme that can be applied in principle to all linearly constrained quadratic and linear programs in bounded integer variables, and the computational findings for a broad spectrum of problem classes raises the possibility that similarly successful results may be obtained for even wider ranges of problems. As our methods for UQP continue to improve with ongoing research, the UQP model offers a representational tool of particular promise.

References

1. Alidaee, B, G. Kochenberger, and A. Ahmadian, "0-1 Quadratic Programming Approach for the Optimal Solution of Two Scheduling Problems," International Journal of Systems Science, 25, 40 1-408, 1994.

2. Alkhamis, T. M., M. Hasan, and M. A. Ahmed, "Simulated Annealing for the Unconstrained Binary Quadratic Pseudo-Boolean Function," *European Journal of Operational Research,* 108, (1998), 641-652.

3. Amini,, M., B. Alidaee, G. Kochenberger, "A Scatter Search Approach to Unconstrained Quadratic Binary Programs," To appear in *New Methods in Optimization,* Come, Dorigo, and Glover, eds., McGraw-Hill, 1999.

4. Beasley, J. E., "Heuristic Algorithms for the Unconstrained Binary Quadratic Programming Problem," Working Paper, Imperial College, 1999.

5. Boros, E, P. Hammer, and X, Sun, "The DDT Method for Quadratic 0-1 Minimization," RUTCOR Research Center, RRR 39-89, 1989.

6. Chardaire, P, and A. Sutter, "A Decomposition Method for Quadratic Zero-One Programming," Management Science, 4 1:4, 704-7 12, 1994.

7. Gallo, G, P. Hammer, and B. Simeone, "Quadratic Knapsack Problems," Mathematical Programming, 12, 132-149, 1980.

8. Glover, F, and M. Laguna, "Tabu Search," Kluwer Academic Publishers, 1997.

9. Glover, F., "A Template for Scatter Search and Path Relinking," School of Business, University of Colorado, Technical Report, February 1998.

10. Glover, F., M. Amini, G. Kochenberger and B. Alidaee, "A New Evolutionary Metaheuristic for the Unconstrained Binary Quadratic Programming: A Case Study of the Scatter Search," School of Business, University of Colorado, Boulder, September 1999.

11. Glover, F., G. Kochenberger, B. Alidaee, and M.M. Amini, "Tabu with Search Critical Event Memory: An Enhanced Application for Binary Quadratic Programs," In: *Meta-Heuristics: Advances and Trends in Local Search Paradigms for Optimization,* (Eds.) S. Voss, S. Martello, I. Osman, and C. Roucairol. Kluwer Academic Publisher, Boston, 1999.

12. Glover, F., G. Kochenberger, B. Alidaee, and M. Amini, "Unconstrained Quadratic Binary Program approach to Quadratic Knapsack Problems," Working paper, Hearin Center for Enterprise Science, University of Mississippi, 1999.

13. Glover, F., B. Alidaee, C. Rego, and G. Kochenberger, " One-Pass Heuristice for Large-Scale Unconstrained Binary Quadratic Programs," Working paper, Hearin Center for Enterprise Science, University of Mississippi, 2001.

14. Glover, F., G. Kochenberger., and B. Alidaee, "Adaptive Memory Tabu Search for Binary Quadratic Programs," Management Science, 44:3, 336-345, 1998.

15. Hammer, P, E. Boros, and X. Sun, "On Quadratic Unconstrained Binary Optimization," *INFORMS National Meeting,* Seattle, October, 1998.

16. Hammer, P., and S. Rudeanu, *Boolean Methods in Operations Research,* Springer-Verlag, New York, 1968.

17. Hansen, P.B., "Methods of Nonlinear 0-1 Programming," *Annals Discrete Math,* vol. 5, pp.53-70, 1979.

18. Hansen, F, B. Jaumard, and V. Mathon, "Constrained Nonlinear 0-1 Programming," INFORMS Journal on Computing, 5:2, 97-119, 1993.

19. Harary, F, "On the Notion of balanced of a signed Graph," Michigan Mathematical Journal, 2, 143-146, 1953/54.

20. Katayama, K., M. Tani, and H. Narihisa, "Solving Large Binary Quadratic Programming Problems by an Effective Genetic Local Search Algorithmn *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO'OO).* Morgan Kaufmann, 2000. (to appear).

21. Kochenberger, G., F. Glover, B. Alidaee, and C. Rego, "Solving Combinatorial Optimization Problems via Reformulation and Adaptive Memory Metaheuristics," *Revolutionary Visions In Evolutionary Computation, ed* Anil Menon and David Goldberg, Kluwer Publisher, (2002).

22. Krarup, J, and A. Pruzan, "Computer Aided Layout Design," Mathematical Programming Study, 9, 75-94, 1978.

23. Laughunn, D.J, "Quadratic Binary programming," Operations Research, 14, 454-461, 1970.

24. Lodi, A., K. Allemand, and T. M. Liebling, "An Evolutionary Heuristic for Quadratic 0-1 Programming," Technical Report OR-97-12, D.E.I.S., University of Bologna, 1997.

25. McBride, R.D., and J.S. Yormack, "An Implicit Enumeration Algorithm for Quadratic Integer Programming," Management Science, 26, 282-296, 1980.

26. Merz, P. and B. Freisleben, "Genetic Algorithms for Binary Quadratic Programming," *Proceedings of the 1999 International Genetic and Evolutionary Computation Conference (GECCO'99),* pp. 4 17-424, Morgan Kaufmann, 1999.

27. Pardalos, P, and G.P. Rodgers, "Computational Aspects of a Branch and Bound Algorithm for Quadratic Zero-one Programming," Computing, 45, 131-144, 1990.

28. Pardalos, P, and G.P. Rodgers, "A Branch and Bound Algorithm for Maximum Clique problem," Computer & OR, 19, 363-375, 1992.

29. Pardalos, F, and J. Xue, "The Maximum Clique Problem," The Journal of Global Optimization, 4, 30 1-328, 1994.

30. Phillips, A.T., and J.B. Rosen, "A Quadratic Assignment Formulation of the Molecular Conformation Problem," Journal of Global Optimization, 4, 229-24 1, 1994.

31. Williams, A. C., "Quadratic 0-1 Programming Using the Roof Duality with Computational Results," Rutcor Research Report 8-85, Rutgers University, New Brunswick, NJ., 1985.

32. Witsgall, C., "Mathematical Methods of site Selection for Electronic System (EMS)," NBS Internal Report, 1975.