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Foundation-Penalty Cuts for Mixed-Integer Programs

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Abstract — We propose a new class of Foundation-Penalty (FP) cuts for GUB-constrained (and ordinary) mixed-integer programs, which are easy to generate by exploiting standard penalty calculations that are routinely employed in branch-and-bound contexts. The FP cuts are derived with reference to a selected integer variable or GUB set, and a foundation function that is typically a reduced cost function corresponding to an optimal linear programming basis. The concept behind the generation of these cuts generalizes the lifting process, and as we demonstrate, bears relationships with other classical cuts such as disjunctive cuts, lift-and-project cuts, convexity cuts, Gomory cuts, and mixed-integer rounding cuts. For example, the easily derived but often useful cutting planes at the level of Gomory cuts and mixed integer rounding cuts are subsumed by related FP cuts that simply 'plug in' penalty values from standard calculations (where the penalties are allowed to go beyond those from the first primitive mixed-integer programming codes). In general, the strength of these FP cuts can be varied according to the trade-offs between the strengths of alternative penalty calculations and the effort required to compute them, by virtue of the interactions between the foundation function and the branching disjunctions that underlie the penalty computations. By this means, FP cuts are especially convenient for use in branch-and-cut procedures, where penalty calculations are employed as a matter of course, and afford new strategies for generating cutting planes in this setting.

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1. Introduction

Consider a mixed-integer zero-one program MIP stated in the form

MIP: Minimize $c^T x$

$$\text{subject to } \ x \in X \equiv X_1 \cap X_2 \tag{1}$$

where $x \in R^n$, X_1 describes a set of constraints representing a polyhedron in R^n, $X_2 \equiv \{x : x_j$ is binary $\forall j \in B \subseteq N\}$, and where $N=\{1,\ldots,n\}$ is the index set of all the variables.

In this paper, we introduce the concept behind a new class of cutting planes for MIP called *Foundation-Penalty cuts* **(FP).** Although we focus on zero-one mixed-integer programs, as discussed in the sequel, many of the ideas extend to general mixed-integer programs as well. As the name suggests, FP cuts are predicated on two elements: a (linear) foundation function, and a set of penalties that are computed based on the conditional values taken on by either a single binary variable, or by several binary variables comprising a *generalized upper bounding* **(GUB)** set. While we discuss both the single integer variable and GUB set cases, it is in the latter context of GUB sets that this class of FP cuts might hold the greatest promise. Previous work that yields special cutting planes for such GUB constrained problems can be found, for example, in Glover (1973, 1975), Hammer et al. (1975), Balas (1979), Sherali et al. (1995), Sherali and Lee (1995), Glover et al. (1997), and Gu et al. (1999).

In fact, as we demonstrate in the sequel, the concept underlying FP cuts generalizes the lifting process introduced by Gomory (1969) and Glover (1970) in the context of group polyhedra, and by Padberg (1973) for 0-1 problems (see also Crowder et al. (1983), Balas and Zemel (1975), Gu et al. (1995), and Nemhauser and Wolsey (1988)). Moreover, the FP cuts bear a relationship to the disjunctive cuts (see Balas et al. (1993, 1998), Sherali and Adams (1990, 1994), and Sherali et al. (1998)), convexity cuts (see Glover (1973)), Gomory cuts (1960a, b), and mixed-integer rounding cuts (see Marchand and Wolsey, (2001)). We explore these relationships in the present paper to afford insights into exploiting the flexibility that is inherent in the class of FP cuts for generating judicious types of cuts. In particular, this flexibility permits the derivation of valid inequalities that cut deeper along specified dimensions as desired.

The remainder of this paper is organized as follows. The next section provides the basic concept underlying the derivation of FP cuts. The relationship of this idea to the lifting process is exposed in Section 3. Certain higher-order extensions of these cuts that consider multiple, non-GUB-related, binary variables are investigated in Section 4. Thereafter, we explore the relationship of FP cuts to several other classical cuts in Section 5, and Section 6 concludes the paper with recommendations for future research in this area.

2. Derivation of the Foundation-Penalty (FP) Cuts

As mentioned in Section 1, the class of FP cuts are governed by two principal elements, namely, a foundation function, and certain penalty computations conditioned on values taken on by either a single binary variable or by a set of GUB-constrained variables. Each of these features that leads to the derivation of the cut is discussed in turn below. (Later in Section 4, we shall extend these cuts to higher-order or multiple variable disjunctions.)

2.1. Foundation Function

The foundation function is some selected linear function of the form $\sum_{j \in J} d_j x_j$, where $J \subseteq N$. Typically, this function might correspond to a reduced cost objective representation associated with some dual feasible solution, or more pertinently, an optimal basis to the linear programming (LP) relaxation $\overline{\text{MIP}}$ of MIP, given by

$$\overline{\textbf{MIP}}: \text{ Minimize } \left\{ c^T x : x \in \overline{X} \right\} \tag{2}$$

where $\overline{X}$ denotes the usual LP relaxation of *X*. In this case, we would have

$$J \equiv \text{\{set of nonbasic variables\}, and } d_j \geq 0 \quad \forall j \in J. \tag{3}$$

2.2. Penalty Characterizations

The penalty computations are conducted with respect to some binary variable $x_k, k \in B$, or with respect to a set of variables that are GUB-constrained according to

$$\sum_{k \in K} x_k = 1, \text{ where } K \subseteq B. \tag{4}$$

(The case of multiple non-GUB restricted binary variables is considered later in Section 4, and the treatment of general integer-variables is addressed in Corollary 1 below. However, for clarity in presentation, we focus on the aforementioned two cases first.)

For any binary variable x_k, $k \in B$, let us define P_{k1} and P_{k0} as lower bounds on the respective values z_{k1} and z_{ko} of MIP under the corresponding additional conditions based on the disjunction that $x_k = 1$ or $x_k = 0$. More specifically,

$$P_{k1} \leq z_{k1} \equiv \text{minimum} \left\{ \sum_{j \in J} d_j x_j : x \in X, \text{and } x_k = 1 \right\}, \text{ and} \tag{5a}$$

$$P_{k0} \leq z_{k0} \equiv \text{minimum} \left\{ \sum_{j \in J} d_j x_j : x \in X, \text{and } x_k = 0 \right\}. \tag{5b}$$

Observe that while LP relaxations afford the most natural mechanism for computing these lower bounding values P_{k1} and P_{k0} in (5), there are a variety of different alternatives by which these quantities might be generated. For example, these values could be based on the simple penalties derived via a single dual simplex pivot on an optimal LP tableau for $\overline{\text{MIP}}$ that has been augmented by the additional restriction $x_k = 1$ or $x_k = 0$, or via multiple dual simplex pivots of this type as used in *strong branching* strategies (see Applegate et al. (1996)). Alternatively, we could solve integer knapsack relaxations based on surrogate constraint strategies (see Rardin and Karwan (1984)). Of course, if any of the penalty computations yield $P_{k1} = \infty$ or $P_{k0} = \infty$, we simply enforce the opposite restriction $x_k = 0$ or $x_k = 1$, respectively, and conduct subsequent implied reductions via standard logical tests (see Nemhauser and Wolsey

(1988)). Hence, in what follows, we will always assume that all penalties derived are finite. Note that in the GUB case, we compute P_{k1} for each $k \in K$, where for any $k \in K$, the computation of P_{k1} in (5a) is conducted by also explicitly enforcing $x_j = 0 \; \forall j \in K - \{k\}$ by virtue of the presence of (4) within the defining set *X*, and similarly, for other GUB sets that contain x_k. Albeit simple, this observation is frequently overlooked in the literature on MIP penalty calculations, yet can have a considerable impact on the penalties generated, particularly in problems where a given x_k belongs to numerous GUB sets. The effect of compelling the indicated accompanying GUB variables to equal zero will automatically be achieved if a penalty calculation is based on performing a sufficient number of dual pivots, but possibly at the expense of undue computational effort. Setting $x_j = 0$ can of course be conveniently handled for any nonbasic x_j simply by disregarding the associated component d_j and its column in performing the penalty calculations. The main result for deriving the FP cuts based on the foregoing constructs can be stated as follows.

Theorem 1. Given a foundation function and penalty computations as defined in (5), the Foundation-Penalty (FP) cut for the case of a single binary variable x_k as given by

$$\sum_{j \in J} d_j x_j \geq P_{k1} x_k + P_{k0}(1 - x_k) \tag{6a}$$

and for the GUB-constrained case (4) as given by

$$\sum_{j \in J} d_j x_j \geq \sum_{k \in K} P_{k1} x_k \tag{6b}$$

yield valid inequalities for MIP. Moreover, in either case, under the condition (3) corresponding to an optimal basis for the LP relaxation $\overline{\text{MIP}}$ of MIP, if any of the penalties are positive for a currently fractional variable x_k in the LP solution, then (6) provides a *separating* inequality that deletes this LP solution.

Proof. The validity of (6a) follows directly from the penalty definitions (5) and the disjunction that $\{x_k = 1$ or $x_k = 0\}$, and that for (6b) follows from (5a) and that $x_k = 1$ for exactly one $k \in K$, and is zero otherwise. Moreover, under the stated condition based on the LP relaxation $\overline{\text{MIP}}$, since the left-hand side of (6) is zero for this LP relaxation solution while the right-hand side is positive, the inequality (6) deletes this LP solution. This completes the proof.

A direct extension of (6a) remains valid for any general integer restricted variable x_k as well, under the following conditions.

Corollary 1. Suppose that the foundation function conforms with (3) corresponding to the LP relaxation solution, and that an integer-restricted variable x_k currently takes on a value b_k that is fractional. Let P_k^+ and P_k^- be the respective values of the LP relaxations derived via

$$P_k^+ = \text{minimum}\left\{\sum_{j \in J} d_j x_j : x \in \overline{X},\; x_k \geq \lfloor b_k \rfloor + 1\right\} \tag{7a}$$

$$P_k^- = \text{minimum}\left\{\sum_{j \in J} d_j x_j : x \in \overline{X}, x_k \leq \lfloor b_k \rfloor\right\}. \tag{7b}$$

Then the following inequality is valid

$$\sum_{j\in J} d_j x_j \geq P_k^+(x_k - \lfloor b_k \rfloor) + P_k^-(\lfloor b_k \rfloor + 1 - x_k). \tag{8}$$

Moreover, (8) is a separating inequality if either $P_k^+ > 0$ or $P_k^- > 0$.

Proof. Note that the disjunction $x_k \geq \lfloor b_k \rfloor + 1$ or $x_k \leq \lfloor b_k \rfloor$ is valid. In the former case, when $x_k = \lfloor b_k \rfloor + 1$, then clearly (8) is valid from (7a). Note that by (3), for the problem

$$v(\theta) \equiv \min\left\{\sum_{j\in J} d_j x_j : x \in \overline{X}, x_k \geq \theta\right\} \tag{9}$$

we have $v(\lfloor b_k \rfloor) = 0$. Since $v(\lfloor b_k \rfloor + 1) = P_k^+$ from (7a), and since $v(\theta)$ is a piecewise linear convex nondecreasing function of θ, we have that

$$\sum_{j\in J} d_j x_j \geq P_k^+(x_k - \lfloor b_k \rfloor)\ \forall\ x_k \geq \lfloor b_k \rfloor + 1, \text{ with } x \in \overline{X}. \tag{10}$$

Noting that the second term in (8) is nonpositive for $x_k \geq \lfloor b_k \rfloor + 1$ (since $P_k^- \geq 0$), we have that (8) is true for this case. Likewise, by a parallel argument, (8) is true for all $x_k \leq \lfloor b_k \rfloor$, thereby establishing the validity of (8). Moreover, if $P_k^+ > 0$ or $P_k^- > 0$, then the right-hand side of (8) is positive when $x_k = b_k$, while the left-hand side of (8) is zero at the current LP solution. Hence, (8) is a separating inequality in this case, and this completes the proof.

Remark 1. The cut (6) (or (8)) can be strengthened by attempting to reduce the coefficients d_j on the left-hand side, while preserving the validity of the penalties defining the right-hand sides. As such, in retrospect, this should be taken into consideration during the process of defining the foundation function and deriving the associated penalties for generating the cut itself, rather than at a subsequent sequential step, whenever possible.

We now proceed to discuss the relationship of FP cuts with other classical cuts, which serves to provide additional insights into the selection of the basic elements defining FP cuts.

3. Relationship with Lifting Concepts

To illustrate this connection, consider the lifting of minimal cover inequalities for knapsack constraints as expounded by Crowder et al. (1983). Given a knapsack constraint of the general form

$$\sum_{j\in N_1} a_j x_j \geq b, \text{ where } 0<a_j\leq b\ \forall j \in N_1, \text{ and where } N_1 \subseteq B, \tag{11}$$

let C be a minimal cover in the sense that

$$\sum_{j\in N_1 - C} a_j < b, \text{but} \sum_{j\in N_1 - C} a_j + \min_{k\in C}\{a_k\} \geq b. \tag{12}$$

Hence,

$$\sum_{j \in C} x_j \geq 1 \tag{13}$$

is a valid (minimal cover) inequality. In order to lift (13) into the dimension of an additional variable x_t, $t \in N_1 - C$, and obtain a valid inequality

$$\sum_{j \in C} x_j \geq 1 + \alpha(1 - x_t), \tag{14}$$

Crowder et al. note that (14) is always valid when $x_t = 1$, while to maintain validity under the condition that $x_t = 0$ requires that

$$(1+\alpha) \leq \text{minimum}\left\{\sum_{j \in C} x_j : \sum_{j \in N_1} a_j x_j \geq b,\ x_j \text{ binary } \forall j \in N_1, \text{ and } x_t = 0\right\}. \tag{15}$$

Observe that this is akin to the derivation of an FP cut using $\sum_{j \in C} x_j$ as the foundation function, and considering the binary knapsack constraint as the set $x \in X$ (or its relaxation) in (5b). Accordingly, we can equate $P_{t0} = (1+\alpha)$. Furthermore, note that when we enforce $x_t = 1$ instead of $x_t = 0$ in the right-hand side of (15), we have from (12) that the objective value of the resulting knapsack problem is 1, so that we know *a priori* that $P_{t1} = 1$. Hence, the FP cut (6a) in this case would be

$$\sum_{j \in C} x_j \geq P_{t1} x_t + P_{t0}(1 - x_t) = x_t + (1+\alpha)(1 - x_t) = 1 + \alpha(1 - x_t),$$

which coincides with (14). The analogy continues in a likewise fashion for subsequent lifting steps in this sequential process. In a similar vein, the FP cuts bear this conceptual relationship with the more general one- and zero-lifting described in Gu et al. (1999), and to the simultaneous-GUB lifting described in Sherali and Lee (1995), where the latter relates to (6b).

4. Higher-order FP Cuts

The FP cut (6a) has been derived with respect to the disjunction concerning a single variable x_k. The simple conceptual form of (6a) readily provides the flexibility of extending this cut to the simultaneous consideration of two or more variables, yielding higher-order FP cuts.

To illustrate, consider the case of a pair of binary variables x_k and x_1. Analogous to (5), let

$$P_{k1}(p,q) \leq z_{k1}(p,q) \equiv \text{minimum}\left\{\sum_{j \in J} d_j x_j : x \in X, \text{and } (x_k, x_1) = (p,q)\right\} \tag{16a}$$

$$\text{for } (p,q) \in V \equiv \{(0,0),(1,0),(0,1),(1,1)\}. \tag{16b}$$

Accordingly, analogous to (6a), we can assert that the following inequality is valid, because precisely one term on the right-hand side is nonzero, and equal to one, for any binary values of x_k and x_l, and then, the associated penalty is valid via (16).

$$\sum_{j\in J} d_j x_j \geq P_{k1}(1,1)x_k x_1 + P_{k1}(1,0)x_k(1-x_1) + P_{k1}(0,1)(1-x_k)x_1 + P_{k1}(0,0)(1-x_k)(1-x_1). \quad (17)$$

Observe that the right-hand side of (17) contains a quadratic product term $x_k x_1$. We can linearize this term by substituting $w_{k1} \equiv x_k x_1$ and accommodating bound-factor products as in Sherali and Adams (1990), for example, to get

$$\sum_{j\in J} d_j x_j \geq \left[P_{k1}(1,1) + P_{kl}(0,0) - P_{k1}(1,0) - P_{k1}(0,1) \right] w_{k1}$$

$$+P_{k1}(1,0)x_k + P_{k1}(0,1)x_1 + P_{k1}(0,0)(1-x_k-x_1) \quad (18a)$$

where

$$w_{k1} \leq x_k, w_{k1} \leq x_1, w_{k1} \geq 0, \text{ and } w_{k1} \geq x_k + x_1 - 1. \quad (18b)$$

Notice that if the coefficient [•] of w_{k1} in (18a) is positive, then by the nature of this inequality, it is only relevant to impose the last two constraints in (18b). Likewise, if this coefficient [•] is negative in (18a), only the first two inequalities in (18b) are relevant. Alternatively, we can use the relevant pair of inequalities from (18b) to project out w_{k1} from (18a) and derive a pair of corresponding valid inequalities having only the original variables x_k and x_1 appearing in the right-hand side of (18a).

5. Relationship of FP Cuts to a Variety of Classical Valid Inequalities

The FP cuts bear a relationship with a variety of classical cuts such as disjunctive cuts, lift-and-project or RLT (reformulation-linearization technique) cuts, convexity cuts, mixed-integer rounding cuts, Gomory cuts, etc. As an illustration, to expose this association in the interesting context of GUB constraints, let us consider the MIP restrictions in the following form, where we have explicitly displayed a particular focal GUB constraint (4) based on a GUB set *K*, and where x_0 represents the vector of variables indexed by *N-K*, which are presently all relaxed to be continuous.

$$Ax_0 + \sum_{k\in K} A_k x_k \geq b \quad (19a)$$

$$\sum_{k\in K} x_k = 1 \quad (19b)$$

$$x_k \text{ binary } \forall\, k \in K, x_0 \geq 0. \quad (19c)$$

Note that while we have considered the constraints in (19a) to be all inequalities for ease in notation, equality constraints can also be included and handled in a similar fashion below. We can construct the convex hull representation of (19) by using the GUB special-structured RLT process described in Sherali et al. (1998). This involves multiplying (19a) and $x_0 \geq 0$ in (19c) by each x_k, $k \in K$, multiplying (19b) by x_0, applying the fact that $x_k^2 = x_k \quad \forall\, k \in K$, and $x_k x_1 = 0\ \forall\, k,1 \in K, k \neq 1$, and then substituting the vector y_k in place of the product term $x_0 x_k, \forall k \in K$. This yields the following representation

$$Ay_k - (b - A_k)x_k \geq 0 \quad \forall \ k \in K \quad (20a)$$

$$x_0 - \sum_{k\in K} y_k = 0 \tag{20b}$$

$$\sum_{k\in K} x_k = 1 \tag{20c}$$

$$x_k \geq 0 \; \forall k \in K, \; y_k \geq 0 \; \forall \; k \in K. \tag{20d}$$

Consequently, any valid inequality for (19) can be obtained (or implied by) the projection of (20) onto the space of the original problem variables $\left(x_0, x_k \text{ for } k \in K\right)$. By LP duality (or Farkas' lemma), all such valid inequalities are obtained as weighted surrogates of (20) that zero out the coefficients for the new variables $y_k, \; \forall k \in K$. In other words, denoting $\pi_k \geq 0, \; \pi,$ and π_0 as the surrogate multipliers (dual variables) associated with (20a), (20b), and (20c), respectively, we obtain that any valid inequality is of the form

$$\pi^T x_0 + \sum_{k\in K} \left[\pi_0 - \pi_k^T\left(b - A_k\right)\right] x_k \geq \pi_0$$

where

$$\pi_k^T A - \pi^T \leq 0, \; \pi_k \geq 0, \forall k \in K.$$

Using the constraint $\sum_{k\in K} x_k = 1,$ this yields any valid inequality in the form

$$\pi^T x_0 \geq \sum_{k\in K} \pi_k^T\left(b - A_k\right) x_k \tag{21a}$$

where

$$\pi_k^T A \leq \pi^T, \pi_k \geq 0 \; \forall \; k \in K. \tag{21b}$$

Note that such cuts as obtained using the special-structured RLT process of Sherali et al. (1998) are a generalization of the lift-and-project cuts of Balas et al. (1993), where the latter are generated in a similar fashion but with respect to a single variable rather than with respect to a GUB set of variables as used above. Observe the relationship between (21a) and (6b). In particular, if we designate $\pi^T x_0$ as the foundation function, we can compute the corresponding penalty in (5a) as

$$P_{k1} = \text{ minimum } \left\{\pi^T x_0 : \; Ax_0 \geq \left(b - A_k\right), x_0 \geq 0\right\}, \forall k \in K, \tag{22}$$

where the constraints of the problem in (22) correspond to fixing $x_k = 1$ in (19). Let π_k^* be an optimal dual multiplier associated with the (structural) constraints in (22), for each $k \in K$. Then we have,

$$P_{k1} = \pi_k^{*T}\left(b - A_k\right), \text{where } \pi_k^{*T} A \leq \pi^T, \text{and } \pi_k^* \geq 0, \forall k \in K. \tag{23}$$

Hence, the associated FP cut (6b) would then be given by

$$\pi^T x_0 \geq \sum_{k\in K} P_{k1} x_k = \sum_{k\in K} \pi_k^{*T}\left(b - A_k\right) x_k \tag{24}$$

which is precisely of the form (21), noting (23). In fact, given any π_k satisfying (21b), $\forall k \in K,$ the corresponding cut (21a) would be dominated by (24), since by duality in (22), $\pi_k^{*T}\left(b - A_k\right) \geq \pi_k^T\left(b - A_k\right)$ for all π_k feasible to (21b).

Cuts of the foregoing type can also be essentially viewed as *disjunctive cuts*. To expose this relationship further in a more general setting, consider the disjunction

$$\{\text{At least one of } A_k x \geq b_k, \; x \geq 0, \text{ must be satisfied, for some } k \in K\}. \tag{25}$$

The basic disjunctive principle of Balas (1979, 1998) and Jeroslow (1977) (see also Glover (1975) and Sherali and Shetty (1980)), portends that any valid inequality for this disjunction can be derived as follows, by associating surrogate multipliers $\pi_k \geq 0$ with the constraints $A_k x \geq b_k, \forall k \in K$:

$$\pi^T x \geq \pi_0, \text{ where } \pi^T \geq \pi_k^T A_k \quad \forall k \in K, \quad \pi_0 \leq \pi_k^T b_k \quad \forall k \in K. \tag{26}$$

Observe that we can imagine that the disjunction (25) relates to a GUB constraint $\sum_{k \in K} y_k = 1$, where for each $k \in K$, the binary variable y_k when put equal to 1 enforces the corresponding constraints $A_k x \geq b_k, x \geq 0$ to hold true. Accordingly, in the context of the FP cut, based on $\pi^T x$ as the foundation function, we can compute penalties via (5a) as

$$P_{k1} = \text{minimum } \left\{ \pi^T x : \; A_k x \geq b_k, x \geq 0 \right\} = \pi_k^{*T} b_k, \forall k \in K \tag{27}$$

where π_k^* is an optional dual solution to (27), $\forall k \in K$. Hence, in particular, we have,

$$\pi_k^{*T} A_k \leq \pi^T, \quad \pi_k^* \geq 0, \quad \forall k \in K .$$

The corresponding FP cut (6b) would then be given as

$$\pi^T x \geq \sum_{k \in K} P_{k1} y_k . \tag{28}$$

This cut implies any valid cut (26) for the given foundation function $\pi^T x$, because from (27), noting by duality that

$$\pi_k^* b_k \geq \pi_k^T b_k \quad \forall \quad \pi_k \geq 0 \text{ such that } \pi_k^T A_k \leq \pi^T , \tag{29}$$

we have,

$$\pi^T x \geq \sum_{k \in K} P_{k1} y_k = \sum_{k \in K} \left(\pi_k^{*T} b_k \right) y_k \geq \sum_{k \in K} \left(\pi_k^T b_k \right) y_k \geq \sum_{k \in K} \pi_0 y_k = \pi_0 .$$

This general relationship with disjunctive cuts extends the relationship of FP cuts to the convexity cuts of Glover (1973) when based on polyhedral convex sets, to Gomory's (mixed) integer cuts (1960b), and to mixed-integer rounding cuts discussed in Marchand and Wolsey (2001) (which are in essence Gomory cuts), all of which are derived based on the formulation of specific disjunctions of the type (25). The ability to produce deeper FP cuts by generating stronger penalties is a particularly useful feature. (For example, the mixed-integer rounding cuts and the Gomory cuts are generated by Corollary 1 for the simple penalty value that was the first to be used in MIP methods.) This rich association with other cutting plane proposals, allowing them to be derived and analyzed by reference to the FP representation, provides insights into the flexibility and latent capability inherent in this class of FP cuts.

6. Concluding Remarks

The Foundation-Penalty (FP) cuts offer a previously unavailable opportunity to exploit penalty calculations of the type customarily used in branch-and-bound, thereby yielding a new utility for these calculations that supplements their role in fathoming nodes and in selecting branches of the branch-and-bound tree. Consequently, the FP cuts are particularly relevant for use in branch-and-cut methods.

The introduction of this new class of cutting planes also opens up several areas of related research. The latitude to select the foundation function in order to bias the cut to extend more deeply in particular dimensions invites an investigation of alternative strategies for generating these functions. Similarly, the trade-offs involved in employing more advanced penalty calculations in the process of generating the cutting planes warrant investigation. In particular, higher-order penalties may provide a different degree of advantage for FP cuts than for branch-and-bound fathoming operations, since the latter are only relevant in the case where it is possible to determine infeasibility or to establish that the objective function exceeds an admissible bound.

The determination of which GUB sets from a given collection provide the best source for FP cuts also invites investigation. Similarly, the selection of single or multiple variables in generating first-order or higher-order FP cuts is an interesting avenue for further research. We anticipate that MIP problems in which GUB constraints are numerous and include a large portion of the integer variables are likely to provide the most useful applications for these cutting planes. The fact that the FP cuts in such settings are based on selecting GUB sets rather than individual variables as a foundation for creating the cutting plane structure imparts them a novel property whose consequences likewise deserve study.

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