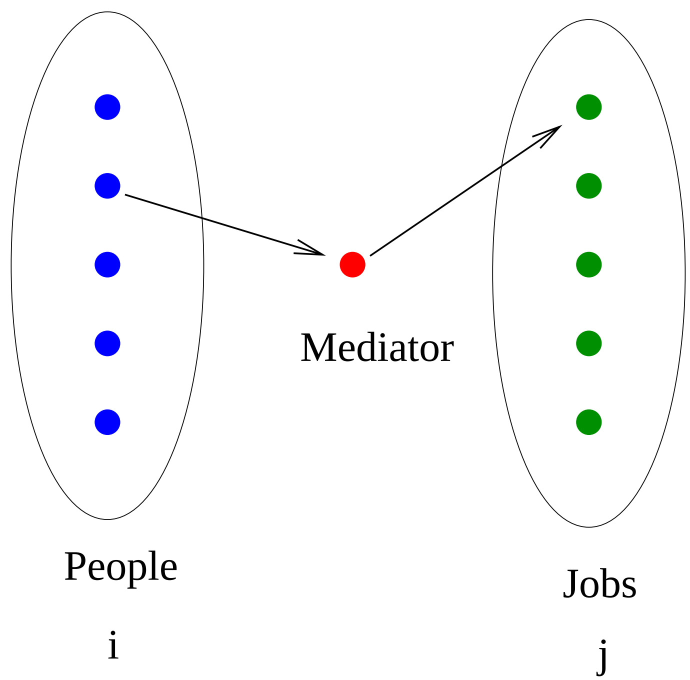
The Optimal Partition and The Assignment Problem

Allen Holder

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

Hearin Center for Enterprise Science
School of Business Administration, Ole Miss

Our Scenario

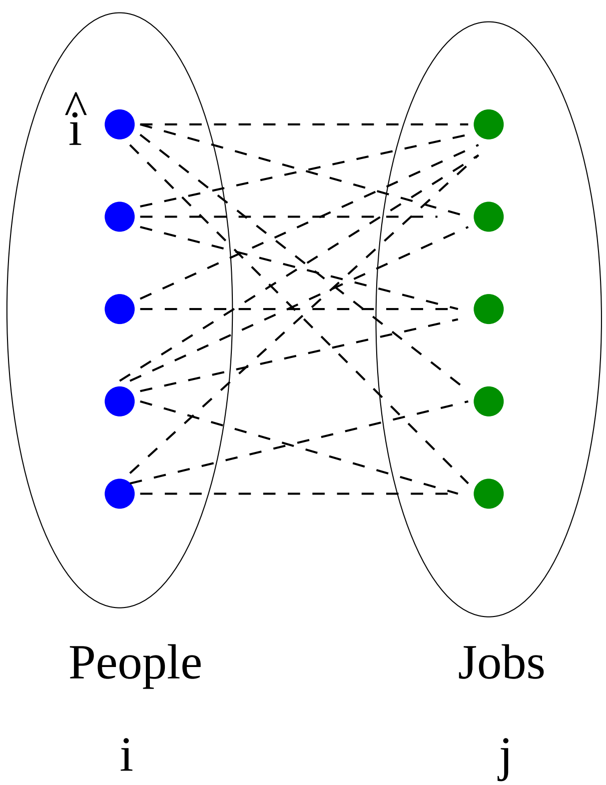


1) The mediator handles all assignments, but does not know who will contact him/her.

2) Jobs are off the market once offered to the candidate.

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

An Assignment Problem



$$\min \sum_{(i,j)} c_{(i,j)} x_{(i,j)}$$

Such That

$$\sum_j x_{(i,j)} \geq \theta, \ i = \hat{i}$$

$$\sum_j x_{(i,j)} \geq 1, \ i \neq \hat{i}$$

$$\sum_i x_{(i,j)} \leq 1, \ \forall j$$

$$x_{(i,j)} \in \{0, 1\}$$

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

The Objective*

The objective function is

$$c_{(i,j)} = T_{(i,j)} + L_{(i,j)} + P_j + G_{(i,j)},$$

where

$$T_{(i,j)} = \begin{cases} 1, & \text{sailor } i \text{ lacks 2 or more NECs required by job } j \\ 0, & \text{otherwise,} \end{cases}$$

$$L_{(i,j)} = \begin{cases} 1, & \text{job } j\text{'s location is different from sailor } i\text{'s current location} \\ 0, & \text{otherwise,} \end{cases}$$

$$P_j = \begin{cases} 1, & \text{job } j\text{'s priority is lower than the average priority} \\ 0, & \text{otherwise,} \end{cases}$$

$$G_{(i,j)} = \begin{cases} 1, & \text{job } j\text{'s location is not preferred by sailor } i \\ 0, & \text{otherwise.} \end{cases}$$

Goals

- Provide person $\hat{\imath}$ with a sufficiently long job list, but recall that the jobs are off the market while person $\hat{\imath}$ contemplates the decision.
- Design job lists in a manner that leaves as much opportunity for subsequent people that call while jobs are off the market.

In the Navy model, list length is based on performance scores and the time to the End-of-Obligated-Service (EOS).

Linear Programming Relaxation

1) The assignment polytope has binary valued extreme points.

and

2) Simplex based algorithms terminate with an extreme point solution.

$\Rightarrow$ We can replace the binary constraint $x_{(i,j)} \in \{0,1\}$ with $0 \leq x_{(i,j)} \leq 1$, and solve with a simplex algorithm.

However, Interior Point algorithms terminate with a different type of solution, which provides a more complete view of the optimal set.

The Optimal Partition

Most interior point algorithms terminate with a solution that induces the Optimal Partition $(B|L|U)$.

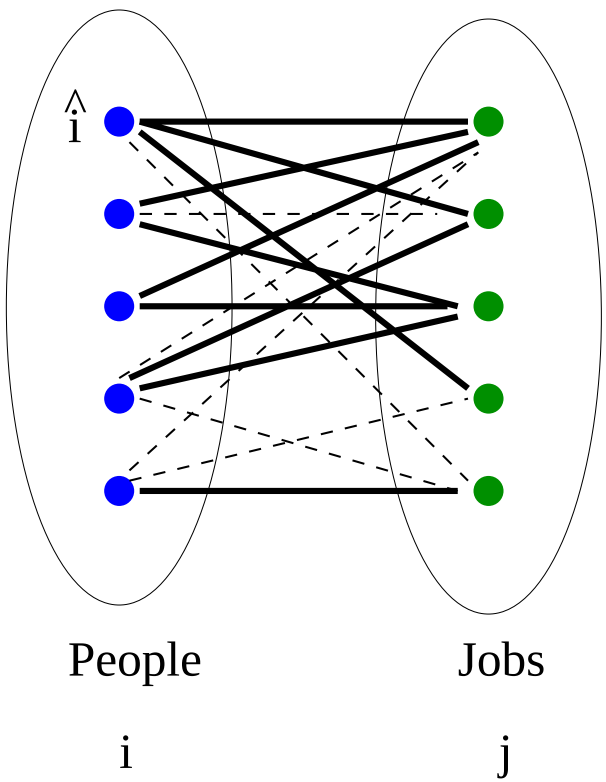
$$\begin{aligned} B &= \{(i,j) : 0 < x^*_{(i,j)} < 1, \text{ for some opt. } x^*\} \\ L &= \{(i,j) : x^*_{(i,j)} = 0, \forall \text{ opt. } x^*\} \\ U &= \{(i,j) : x^*_{(i,j)} = 1, \forall \text{ opt. } x^*\} \end{aligned}$$

Theorem For the assignment polytope we have that

$$\begin{aligned} (i,j) \in B &\Rightarrow \text{Assignment } (i,j) \text{ is sometimes optimal} \\ (i,j) \in L &\Rightarrow \text{Assignment } (i,j) \text{ is never optimal} \\ (i,j) \in U &\Rightarrow \text{Assignment } (i,j) \text{ is always optimal} \end{aligned}$$

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

Visualizing the Optimal Partition

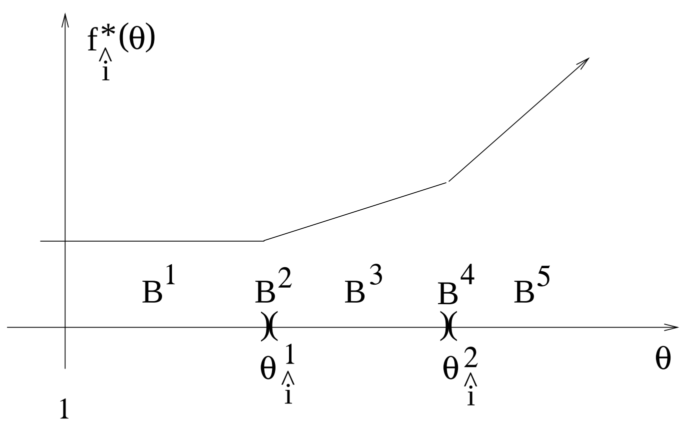


The dotted edges are possible assignments that are never optimal. The solid edges are optimal assignments in some optimal matching.

We want to increase θ as much as possible so that this structure is retained.

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

Parametric Analysis



The optimal subgraph changes at $\theta^1_{\hat{i}}$ and $\theta^2_{\hat{i}}$. Each subgraph is optimal for the corresponding θ value. Polynomial time Algorithms to calculate $\theta^k_{\hat{i}}$ and the optimal partitions are known[a,b].

Theorem The values of $\theta^k_{\hat{i}}$ are integer valued.

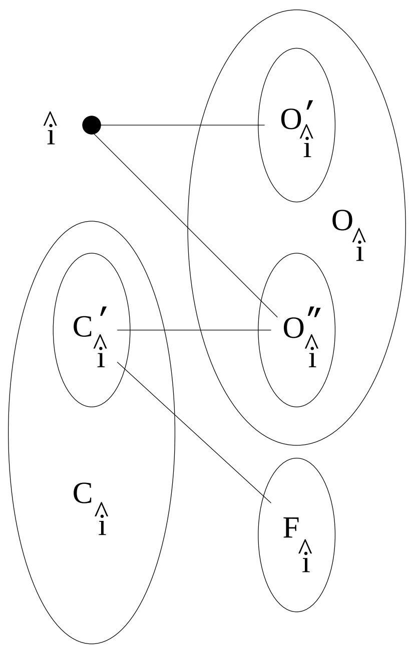
---

[a] I.Adler and R. Monteiro, *A geometric view of parametric linear programming*, Algorithmica, vol. 8, pp. 161-176, 1992.

[b] C. Roos, T. Terlaky, and J. -Ph. Vial, *Theory and Algorithms for Linear Optimization: An Interior Point Approach*, John Wiley & Sons, 1997.

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

Interpreting the First Break Point



$O_{\hat{i}}$ – optimal jobs for person $\hat{i}$

$O'_{\hat{i}}$ – uniquely optimal jobs for person $\hat{i}$

$O''_{\hat{i}}$ – jobs optimal for person $\hat{i}$ and competitors who have another uniquely optimal job.

Theorem We have that $|O'_{\hat{i}}| + |O''_{\hat{i}}| \leq \theta^1_{\hat{i}} \leq |O_{\hat{i}}|$.

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

Interpreting the First Break Point

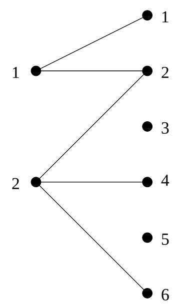
Let $G^* = (P, J, E^*)$ be the optimal subgraph. Define $K(G^*)$ by

$$K(G^*) = \min\{|N(W)| - |W| : W \subseteq P\}.$$

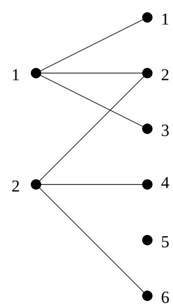
Theorem We have that $\theta^1_{\hat{i}} \geq K(G^*) + 1$.

From the last two results, we see that the first break point estimates competition for person $\hat{i}$ and for the most heavily competitive portion of the graph.

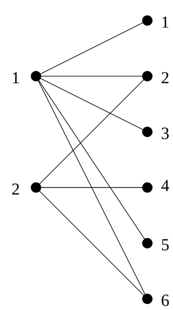
The Optimal Partition and The Assignment Problem



The optimal solutions for $\theta = 1$. We have that $\mathcal{J}_1^0 = \{1, 2\}$, and the current list is $\mathcal{L} = (1, 2)$.

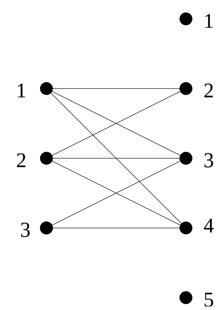


The optimal solutions for $\theta = 2$. Since job 3 is now optimal for person 1, we have that $\mathcal{J}_1^1 = \{3\}$. The list is now $\mathcal{L} = (1, 2|3)$.

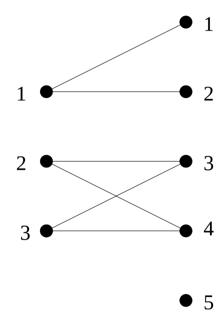


The optimal solution for $\theta = 3$. Jobs 5 and 6 are now optimal for person 1, and $\mathcal{J}_1^2 = \{5, 6\}$. The final list is $\mathcal{L} = (1, 2|3|5, 6)$.

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder



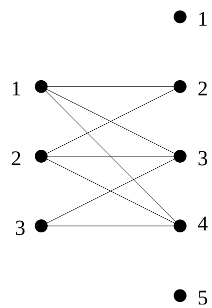
The optimal solutions for $\theta = 1$. Since jobs 2, 3, and 4 are optimal for person 1, $\mathcal{J}_1^0 = \{2, 3, 4\}$. The current list is $\mathcal{L} = (2, 3, 4)$



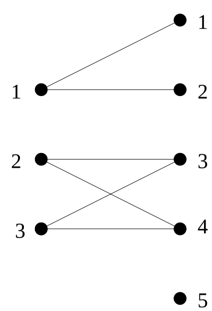
Job 1 becomes optimal and is adjoined to the list: $\mathcal{L} = (2, 3, 4|1)$.

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

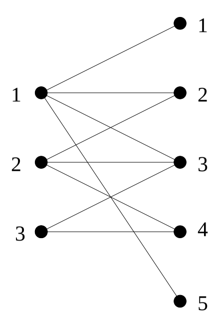
The Optimal Partition and The Assignment Problem



The optimal solutions for $\theta = 1$ form the master plan. Since no jobs are uniquely optimal for person 1, $\mathcal{J}_1^0 = \emptyset$.



The optimal solutions for $\theta = 2$. Since job 2 is optimal for person 1 here and in the master plan, $\mathcal{J}_1^1 = \{2\}$. Since job 1 is is not optimal in the master plan, $\mathcal{J}_1^2 = \{1\}$. The current job list is $\mathcal{L} = (2|1)$.



The optimal solutions for $\theta = 3$. Jobs 3 and 5 are now optimal for sailor 1. Since job 3 is in the master plan and job 5 is not, we have that $\mathcal{J}_1^3 = \{3\}$ and $\mathcal{J}^4 = \{5\}$. The final list is $\mathcal{L} = (2|1|3|5)$.

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

The End

Please Ask Questions

Allen Holder
aholder@olemiss.edu
http://www.olemiss.edu/~aholder

Hearin Center for Enterprise Science
School of Business Administration, Ole Miss